On the accelerated observer's proper coordinates

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Abstract

Physicists have been interested in accelerated observers for quite some time. Since the advent of special relativity, many authors have tried to understand these observers in the framework of Minkowski spacetime. One of the goals is to right the Minkowskian metric in terms of the proper coordinate system of the accelerated observers. In this paper, I write the metric in terms of the Frenet-Serret curvatures and the proper coordinate system of a general accelerated observer. I test this approach for two well-known observers, namely, the Rindler and the rotating ones, and use it to create a set of observers who rotates while accelerating along the axis of rotation.

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I. INTRODUCTION

Accelerated observers in Minkowski spacetime has been widely studied in physics and there is no doubt about their importance to modern physics. They have been used to study quantum phenomena, like the Unruh effect [1], to extend special relativity to noninertial frames and to understand some properties of general relativity [2, 3]. Some very nice papers on the subject have been published so far [1–12], some of them trying to answer fundamental questions such as "how do electric charges behave in an accelerated frame?" [11, 12]. Here, I do not go into these fundamental issues; rather, I use the tetrad formalism to obtain an expression for the metric tensor in terms of the proper coordinate system of an accelerated observer, Sec. III. I also write the metric tensor in terms of the curvatures of the observer's curve and, in Sec. IV, test the resultant metric for two well-known set of observers: the Rindler and the rotating observers. The procedure is very general and the final form of the metric can be used to obtain the metric tensor in terms of the proper coordinate of any accelerated observer. As an example, I create a new set of observers in Sec. V. A brief introduction to the Frenet-Serret tetrad is given in Sec. II.

Throughout this paper capital Latin letters represent tetrad indices, which run over (0)-(3), while the Greek ones represent coordinate indices, which run over 0-3; the small Latin letters run over 1-3. The frame is denoted by e_A, and its components in the coordinate basis ∂_μ is represented by ${e_A}^\mu$.

II. FRENET-SERRET TETRAD

Let $x^\mu(s)$ be a curve in Minkowski spacetime, where s is its arc length. In this spacetime, Frenet-Serret basis can be defined through the formulas

$$\frac{d{e_{(0)}}^\mu}{ds} = k_1 {e_{(1)}}^\mu, \tag{1}$$

$$\frac{d{e_{(1)}}^\mu}{ds} = k_1 {e_{(0)}}^\mu + k_2 {e_{(2)}}^\mu, \tag{2}$$

$$\frac{d{e_{(2)}}^\mu}{ds} = k_3 {e_{(3)}}^\mu - k_2 {e_{(1)}}^\mu, \tag{3}$$

$$\frac{d{e_{(3)}}^\mu}{ds} = -k_3 {e_{(2)}}^\mu, \tag{4}$$

where ${e_{(0)}}^{\mu} = dx^{\mu}/ds$, and e^{μ}_{A} are the components of the vectors in the Cartesian coordinate basis ∂_{μ} (for a general version of these formulas, that is, a version that holds for either a general coordinate system or a curved spacetime, see p. 74 of Ref. [13]). The functions k_1, k_2 and k_3 are known as first, second and third curvatures, respectively. The curvature k_1 measures how rapidly the curve pulls away from the tangent line at s, while k_2 and k_3 measure, respectively, how rapidly the curve pulls away from the plane formed by $e_{(0)}, e_{(1)}$ and from the hyperplane formed by $e_{(0)}, e_{(1)}, e_{(2)}$ at s (for more details, see Ref. [14]).

It is important to note that when k_1 is zero, only $e_{(0)}$ is defined by the previous formulas. To keep the geometrical meaning of k_2 and k_3, we have to set them equal to zero. In this case, the vectors $e_{(i)}$ must be constant. The same happens with k_3 if k_2 vanishes. However, if we are not worried about the meaning of k_i, we can choose the vectors that are not fixed by these formulas as we wish; of course, they have to satisfy the requirements to be a tetrad basis.

III. THE PROPER COORDINATE SYSTEM OF AN ACCELERATED OBSERVER

In this section, I consider the worldline of two distinct observers and choose a frame that is attached to one of them. After that, I impose the condition needed to ensure that we are using the proper coordinate system of the chosen accelerated observer.

To begin with, let two observers n and o describe the curves $x^{\mu}_{n}(s_n)$ and $x^{\mu}_{o}(s_o)$ in an inertial frame of reference I (see figure 1). Now, let $\Lambda(s_n)$ be a local Lorentz transformation from I to another inertial frame that, in an instant s_n/c (c is the speed of light), coincides with a noninertial frame S attached to the observer n. In searching for the proper coordinate system of n, we want the following to hold:

$$(x^{\nu}_{o} - x^{\nu}_{n}){\Lambda^{0}}_{\nu}(s_n) = 0, \tag{5}$$

that is, both events $x^{\mu}_{n}(s_n)$ and $x^{\mu}_{o}(s_o)$ are simultaneous in the frame S. In what comes next, it is more suitable to use a different approach. Instead of dealing with coordinates directly, I shall deal with vectors first; then, when necessary or convenient, I use coordinates.

As figure 1 suggests, the relation among the vectors $\mathbf{r}_n$, $\mathbf{r}_o$ and $\mathbf{r}$ is

$$\mathbf{r} = \mathbf{r}_o - \mathbf{r}_n. \tag{6}$$

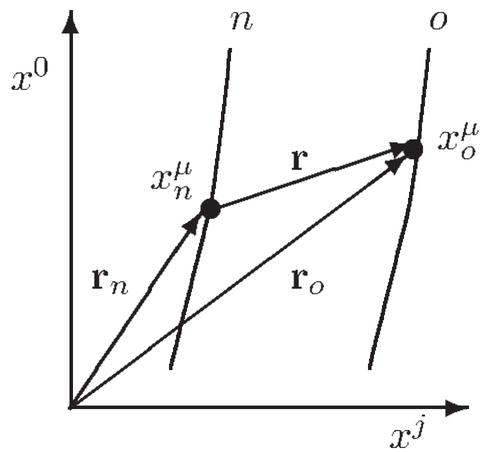

FIG. 1. This figure shows two observers at x_n^μ, x_o^μ and their respective worldline, n and o. The 4-vectors $\mathbf{r}_n$ and $\mathbf{r}_o$ represents the events x_n^μ and x_o^μ, respectively; the 4-vector $\mathbf{r}$ is the position of x_o^μ relative to x_n^μ. Here, x^j represents the axis x, y and z.

Let $\{e_A\}$ be the frame S, which follows the observer n. Since Minkowski spacetime is a flat manifold, the Cartesian vectors ∂_μ are the same at all points of it, which allows us to use the right hand side of $\partial_\mu = e^A{}_\mu(s_n)e_A(s_n)$ at any point; here, the $e^A{}_\mu$'s are the components of the dual basis of e_A written in terms of dx^μ, which can be denoted by $e^A = e^A{}_\mu dx^\mu$. In short, we have $\mathbf{r}_o = x_o^\mu \partial_\mu = x_o^\mu e^A{}_\mu e_A$ and $\mathbf{r}_n = x_n^\mu \partial_\mu = x_n^\mu e^A{}_\mu e_A$. Using these expressions and the definition $\mathbf{r} = r^A e_A$ in Eq. (6), we may write

$$r^A = (x_o^\mu - x_n^\mu)e^A{}_\mu. \tag{7}$$

In this approach the equivalent version of (5) is

$$r^{(0)} = (x_o^\mu - x_n^\mu)e^{(0)}{}_\mu = 0, \tag{8}$$

where $e^A{}_\mu$ plays the role of the local Lorentz transformation, and it was assumed that $e_{(0)}{}^\mu \equiv dx_n^\mu/ds_n$.

By using (8), we can invert (7) to get

$$x_o^\nu = x_n^\nu + r^{(j)}e_{(j)}{}^\nu. \tag{9}$$

The reason why I am writing this is because x_n^ν, and consequently $e_{(j)}{}^\nu$, is supposed to be known and we want to know how the observer o is described in the frame S; the pair $(s_n, r^{(i)})$ will represent the observer o in S.

From (9), we can create a set of rigid observers by taking $r^{(j)}$ constant. For instance, if we take $r^{(i)} = 0$, we have the observer n; for any other value, we have another observer who is at a fixed proper distance from the observer n. The idea is simple: if we have a rigid body, like a rod or a disk, and accelerate it, then x_o^ν is a point in the body (observer) being described by the proper coordinates of the observer who is at the origin of S, that is, the observer n. However, for a general observer o, not necessarily at a fixed distance from n, we can see x_o^ν in (9) as a function of τ and r^j, where $\tau = \tau_n = s_n/c$. Therefore, differentiation of (9) leads to

$$dx_o^\nu = \left(\frac{dx_n^\nu}{d\tau} + r^{(j)} \frac{d{e_{(j)}}^\nu}{d\tau} \right) d\tau + {e_{(j)}}^\nu dr^{(j)}. \tag{10}$$

By using $ds^2 = \eta_{\mu\nu} dx_o^\mu dx_o^\nu$, we get

$$ds^2 = \left[c^2 + 2cr^{(i)} e_{(0)\mu} \frac{d{e_{(i)}}^\mu}{d\tau} + r^{(i)} r^{(j)} \eta_{\mu\nu} \frac{d{e_{(i)}}^\nu}{d\tau} \frac{d{e_{(j)}}^\mu}{d\tau} \right] d\tau^2 + 2r^{(i)} e_{(j)\nu} \frac{d{e_{(i)}}^\nu}{d\tau} d\tau dr^{(j)} + \eta_{(i)(j)} dr^{(i)} dr^{(j)}, \tag{11}$$

where $r^{(1)}$, $r^{(2)}$, and $r^{(3)}$ are such that

$$c^2 + 2cr^{(i)} e_{(0)\mu} \frac{d{e_{(i)}}^\mu}{d\tau} + r^{(i)} r^{(j)} \eta_{\mu\nu} \frac{d{e_{(i)}}^\nu}{d\tau} \frac{d{e_{(j)}}^\mu}{d\tau} > 0. \tag{12}$$

It is important to keep in mind that (11) holds only if e_A and e^A is written in terms of the Cartesian basis ∂_{ct}, ∂_x, ∂_y and ∂_z.

It is clear in (11) that τ, $r^{(i)}$ are the proper coordinates of n. To see this, we just need to set $\tau = constant$ and verify that $ds^2 = \eta_{(i)(j)} dr^{(i)} dr^{(j)}$. Of course, by definition, $ds_n = cd\tau$ ($r^{(i)} = 0$). However, it is interesting to note that although the proper distances of both n and o are the same, the proper time of an observer o at a fixed proper distance from the observe n is not τ, but rather

$$c^2 d\tau_o^2 = \left[c^2 + 2cr^{(i)} e_{(0)\mu} \frac{d{e_{(i)}}^\mu}{d\tau} + r^{(i)} r^{(j)} \eta_{\mu\nu} \frac{d{e_{(i)}}^\nu}{d\tau} \frac{d{e_{(j)}}^\mu}{d\tau} \right] d\tau^2. \tag{13}$$

A. The metric and the curvatures of the observer's curve

Here, I write the line element (11) in terms of the curvatures of the curve described by the observer n, although the interpretation of k_i as the curvatures of this observer's worldline cannot always be true, as we shall see in subsection IV B.

By choosing the basis e_A to be the Frenet-Serret basis (see Eqs. (1)-(4)), the line element (11) can be written as

$$ds^2 = \left[(1+k_1 r^{(1)})^2 - (k_3^2+k_2^2)(r^{(2)})^2 - (k_2 r^{(1)} - k_3 r^{(3)})^2\right] c^2 d\tau^2 + 2\Big[-k_2\delta_{j2} r^{(1)} \\ +(k_2\delta_{j1} - k_3\delta_{j3}) r^{(2)} + k_3\delta_{j2} r^{(3)}\Big] c d\tau dr^{(j)} - \delta_{ij} dr^{(i)} dr^{(j)}, \quad (14)$$

where

$$(1+k_1 r^{(1)})^2 - (k_3^2+k_2^2)(r^{(2)})^2 - (k_2 r^{(1)} - k_3 r^{(3)})^2 > 0. \quad (15)$$

In $2+1$ dimensions, we have $k_3 = 0$ (see [14] for more details). Hence, the line element (14) reduces to

$$ds^2 = \left\{(1+k_1 r^{(1)})^2 - k_2^2\left[(r^{(2)})^2 + (r^{(1)})^2\right]\right\} c^2 d\tau^2 \\ +2k_2\Big[\delta_{j1} r^{(2)} - \delta_{j2} r^{(1)}\Big] c d\tau dr^{(j)} - \delta_{ij} dr^{(i)} dr^{(j)}. \quad (16)$$

In the next section, I use (16) to obtain the line element for the Rindler and the rotating observers.

IV. PARTICULAR CASES

A. Rindler Observers

The Rindler observers in $1+1$ can be represented by (see, e.g., Ref. [10])

$$x^0 = \left(\frac{c^2+\xi a}{a}\right)\sinh\left(\frac{a\tau}{c}\right), \quad (17)$$

$$x^1 = \left(\frac{c^2+\xi a}{a}\right)\cosh\left(\frac{a\tau}{c}\right), \quad (18)$$

where $a = constant$ is the 4-acceleration of the observer who is at the origin of S (the observer n). From Eq. (1), we get $k_1 = a/c^2$ and $k_2 = k_3 = 0$. For this case, the line element (16) reduces to the well-known expression

$$ds^2 = (1+\frac{a}{c^2}\xi)^2 c^2 d\tau^2 - d\xi^2, \quad (19)$$

where $\xi \equiv r^{(1)}$. Condition (15) implies $\xi \in (-c^2/a, \infty)$.

B. Rotating observers

Let a point that is rotating with angular velocity ω and at a distance R from the origin of a inertial frame I have the coordinates

$$x_n^0 = ct, \tag{20}$$

$$x_n^1 = R\cos\omega t, \tag{21}$$

$$x_n^2 = R\sin\omega t. \tag{22}$$

By using the Frenet-Serret basis, we obtain

$$e_{(0)}{}^{\mu} = \gamma(1, -\frac{\omega R}{c}\sin\omega t, \frac{\omega R}{c}\cos\omega t), \tag{23}$$

$$e_{(1)}{}^{\mu} = (0, -\cos\omega t, -\sin\omega t), \tag{24}$$

$$e_{(2)}{}^{\mu} = \gamma(-\frac{\omega R}{c}, \sin\omega t, -\cos\omega t), \tag{25}$$

$$e_{(3)}{}^{\mu} = (0, 0, 0, 1), \tag{26}$$

where $\gamma = 1/\sqrt{1-\omega^2R^2/c^2}$. The curvatures are $k_3 = 0$, and

$$k_1 = \gamma^2\frac{\omega^2}{c^2}R, \tag{27}$$

$$k_2 = \gamma^2\frac{\omega}{c}. \tag{28}$$

Substitution of k_1 and k_2 into (16) gives

$$ds^2 = \left[(1+\gamma^2\frac{\omega^2 R}{c^2}\xi)^2 - \gamma^4\frac{\omega^2}{c^2}(\xi^2+\chi^2)\right]c^2d\tau^2 + 2\gamma^2\omega(\chi d\xi - \xi d\chi)d\tau - d\xi^2 - d\chi^2, \tag{29}$$

where $\xi \equiv r^{(1)}$ and $\chi \equiv r^{(2)}$.

For simplicity, let us set $R = 0$. In this case we have

$$ds^2 = \left[1 - \frac{\omega^2}{c^2}(\xi^2+\chi^2)\right]c^2d\tau^2 + 2\omega(\chi d\xi - \xi d\chi)d\tau - d\xi^2 - d\chi^2, \tag{30}$$

where condition (15) leads to $\sqrt{\xi^2+\chi^2} < c/\omega$. This is the same line element of the rigid disk in Ref. [15].

Here, we have to be very careful with the meaning of k_i. When k_1 is zero the Serret-Frenet formulas do not determine k_2, k_3, $e_{(2)}$ and $e_{(3)}$, as pointed out before. This is exactly the case when one sets $R = 0$ in Eqs. (20)-(22) before evaluate the basis. On the other hand, when we perform the calculations first and then take $R = 0$ we obtain $k_1 = k_3 = 0$ and

$k_2 = \omega/c$, and the basis remains well defined. The problem in this case is that k_i cannot be interpreted as the curvatures of a curve, since a curve which does not curve ($k_1 = 0$) cannot twist ($k_2 \neq 0$). For our purpose this is irrelevant, since we do not need k_i to be curvatures. But now we have the question: "why don't we take the inertial frame, since it also satisfies Frenet-Serret formulas?" The answer is simple: the rotating observers who are not at the origin (ξ *or* $\chi \neq 0$) must be at rest with respect to the chosen frame so that they keep their rigidity in this frame. To understand better, consider the following. If we have just one particle at the origin, then we have two types of frame that the particle can be at rest: a frame that rotates around its origin, and a frame that does not rotate at all. However, if we have a rigid disk made of particles that are at rest with respect to a certain frame, there will be only one frame satisfying this condition: the one which rotates together with the particles. Therefore, we have to choose that frame for the rotating disk.

1. Geodesic equations

The geodesic equations for (30) can be easily solved. It is straightforward to verify that

$$\xi = (C_1 + C_2\lambda)\cos(\omega\alpha\lambda), \tag{31}$$

$$\chi = -(C_1 + C2\lambda)\sin(\omega\alpha\lambda), \tag{32}$$

$$\tau = \alpha\lambda + \tau_0 \tag{33}$$

is the general solution, where C_1, C_2, α and τ_0 are constant, and λ is an affine parameter. The type of curve is determined by

$$\alpha^2 - \frac{C_2^2}{c^2} = \varepsilon, \tag{34}$$

where $\varepsilon = 1$ for timelike, $\varepsilon = -1$ for spacelike, and $\varepsilon = 0$ for null geodesics.

It is interesting to note that the light rays as seen by these observers are not straight lines, but rather spirals. In the next subsection, I evaluate the spectral shift of these light rays.

2. Spectral shift

Let an observer A be at the origin of S and, at $\tau = \tau_1$, send a light signal along the negative ξ-axis to another observer B at ξ_B, χ_B, which receives this signal at τ_2. From

(31)-(33), we find that the light path is

$$x^\mu_{12} = (c\tau, c(\tau_1 - \tau)\cos\omega(\tau - \tau_1), -c(\tau_1 - \tau)\sin\omega(\tau - \tau_1)), \tag{35}$$

where the condition $\sqrt{\xi^2 + \chi^2} < c/\omega$ leads to $\tau \in [\tau_1, 1/\omega + \tau_1)$.

Now, let x^μ_{34} be the path of a signal which is emitted by the observer A at the instant τ_3 and received by B at τ_4. This path can be written as

$$x^\mu_{34} = (c\tau, c(\tau_3 - \tau)\cos\omega(\tau - \tau_3), -c(\tau_3 - \tau)\sin\omega(\tau - \tau_3)), \tag{36}$$

where $\tau \in [\tau_3, 1/\omega + \tau_3)$. Since both paths, (35) and (36), passes by ξ_B, χ_B at τ_2 and τ_4, respectively, then we have (see figure 2)

$$\sqrt{\xi^2_B + \chi^2_B} = c(\tau_2 - \tau_1) = c(\tau_4 - \tau_3). \tag{37}$$

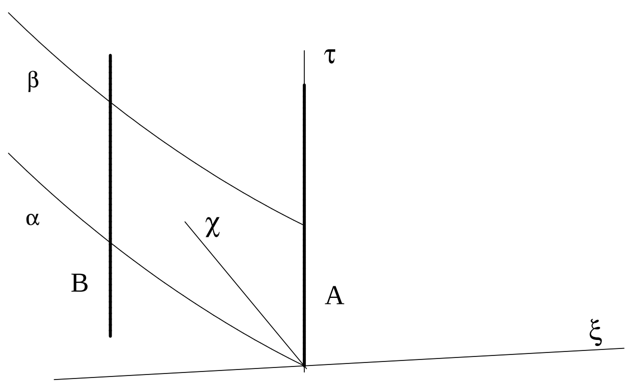

FIG. 2. This figure shows the paths of the observers A and B, the thickest ones, and the paths of the light signals, α and β; the light signals were emitted along the negative ξ-axis. The numerical values in this figure are $\tau_1 = 0$, $c = 1$, $\omega = 1.00 \times 10$ rad/m, $\xi = -3.15 \times 10^{-2}$ m and $\chi = 1.09 \times 10^{-2}$ m. For this case we have $\nu = 1.06\,\nu_0$ (see Eq. (39)).

Suppose that the two light signals represent the wave crests of the same light, with these wave crests being the closest to each other. In this case we may define $\tau_3 - \tau_1 = 1/\nu_0$, where ν_0 is the frequency of light as measured by the observer A. By using Eq. (37) and the line element (30) we define

$$1/\nu \equiv \Delta s_B/c = \left[1 - \frac{\omega^2}{c^2}(\xi^2_B + \chi^2_B)\right]^{1/2} (\tau_4 - \tau_2) \tag{38}$$

and arrive at

$$\nu = \frac{\nu_0}{\sqrt{1 - (\xi^2_B + \chi^2_B)\omega^2/c^2}}, \tag{39}$$

where ν is the frequency measured by the observer B. As one can see, this is a blueshift. This result is in agreement with the Doppler shift formula, which is $\nu = \gamma(\nu - \mathbf{k}\cdot\mathbf{v}/2\pi)$, where $\mathbf{k}$ is the wave vector and $\mathbf{v}$ is the velocity of the observer B, which is zero in S. If we use the Doppler shift formula in the inertial frame I, the vector $\mathbf{v}$ will be perpendicular to $\mathbf{k}$ since, in this frame, the light signals (crest waves) emitted by the observer A will follow radial straight lines. It is important to emphasize that, in the inertial frame I, the source is rotating with the same angular speed ω as the observer B.

V. A NEW SET OF ACCELERATED OBSERVERS

One of the advantages of using (14) is that we can easily construct the line element of any set of observers who mimic rigid bodies. From (9), we are also able to write the equation of these observers without any difficulty. As an example, let the observer n describe the following path in the inertial frame I:

$$x_n^\mu = \sqrt{3}\left(\frac{\sqrt{2}c^2}{a}\sinh\theta, \frac{c^2}{a}\cos\theta, \frac{c^2}{a}\sin\theta, \frac{\sqrt{2}c^2}{a}\cosh\theta\right), \tag{40}$$

where $\theta = as/(\sqrt{3}c^2)$ (s is the arc length), and a is the observer's 4-acceleration. This observer not only rotates around z-axis, but also translate along it (see figure 3).

From the coordinates x_n^1, x_n^2 and the observer's proper time, we can see that the observer (40) rotates around the z-axis with a constant angular speed from his point of view. By equating θ to $2m\pi$ ($m = 0, 1, 2...$), we get $c\tau_m = s_m = 2m\pi\sqrt{3}c^2/a$, which yields the period $T = 2\sqrt{3}c\pi/a$. However, from the point of view of the observer who is at rest in the I frame, this is not a periodic rotation because the function "sinh" is not periodic ($t_{m+1} - t_m$ depends on m).

By using the Frenet-Serret formulas, one obtains the tetrad

$$e_{(0)}{}^\mu = \left(\sqrt{2}\cosh\theta, -\sin\theta, \cos\theta, \sqrt{2}\sinh\theta\right), \tag{41}$$

$$e_{(1)}{}^\mu = \frac{1}{\sqrt{3}}\left(\sqrt{2}\sinh\theta, -\cos\theta, -\sin\theta, \sqrt{2}\cosh\theta\right), \tag{42}$$

$$e_{(2)}{}^\mu = \frac{1}{\sqrt{2}}\left(-\sqrt{2}\cosh\theta, 2\sin\theta, -2\cos\theta, -\sqrt{2}\sinh\theta\right), \tag{43}$$

$$e_{(3)}{}^\mu = \frac{1}{\sqrt{3}}\left(\sinh\theta, \frac{2}{\sqrt{2}}\cos\theta, \frac{2}{\sqrt{2}}\sin\theta, \cosh\theta\right), \tag{44}$$

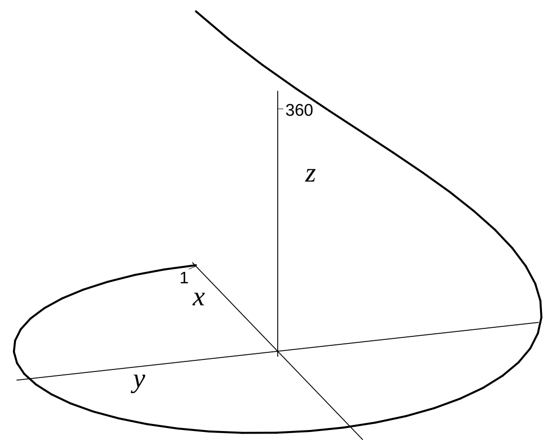

FIG. 3. In this figure, the projection of the curve (40) on x, y, z is shown for $\sqrt{3}c^2/a = 1$ and θ starting from 0.

and the curvatures

$$k_1 = \frac{a}{c^2}, \quad k_2 = \frac{2\sqrt{2}}{3}\frac{a}{c^2}, \quad k_3 = \frac{1}{3}\frac{a}{c^2}. \tag{45}$$

Substitution of (45) into (14) yields

$$\begin{aligned} ds^2 = &\left[\left(1+\frac{a\xi}{c^2}\right)^2 - \frac{a^2}{9c^4}\left(2\sqrt{2}\xi - \lambda\right)^2 - \frac{a^2}{c^4}\chi^2\right] c^2 d\tau^2 \\ &+ \frac{2a}{3c}\left[2\sqrt{2}(\chi d\xi - \xi d\chi) + \lambda d\chi - \chi d\lambda\right] d\tau \\ &- d\xi^2 - d\chi^2 - d\lambda^2, \end{aligned} \tag{46}$$

where $\xi = r^{(1)}$, $\chi = r^{(2)}$ e $\lambda = r^{(3)}$. From (9), we get

$$\begin{aligned} x_o^0 &= A\sinh\theta - \sqrt{2}C\cosh\theta, \quad x_o^1 = B\cos\theta + 2C\sin\theta, \\ x_o^2 &= B\sin\theta - 2C\cos\theta, \quad x_o^3 = A\cosh\theta - \sqrt{2}C\sinh\theta, \end{aligned} \tag{47}$$

where

$$A = \sqrt{6}\left(\frac{c^2}{a} + \frac{\xi}{3} + \frac{\lambda}{3\sqrt{2}}\right), \quad B = \sqrt{3}\left(\frac{c^2}{a} - \frac{\xi}{3} + \frac{2\lambda}{3\sqrt{2}}\right), \quad C = \frac{\chi}{\sqrt{2}}. \tag{48}$$

One can easily verify that this curve satisfies $(x_o^0)^2 - (x_o^1)^2 - (x_o^2)^2 - (x_o^3)^2 = -(A^2 + B^2 + 2C^2)$. By taking ξ, χ and λ as constant, we obtain a set of observers who are at rest in the S frame. Perhaps, these observers may be interpreted as a particular case of a solid cylinder that rotates around its axis and, at the same time, is accelerated along it.

VI. FINAL REMARKS

The proper coordinate system used in (11) belongs to the observer n and, in this sense, this observer is privileged. This is not a strange fact because, in general, each observer has a different 3-velocity with respect to the frame I. For instance, the magnitude of the 3-velocity of a Rindler observer can be shown to be

$$V = \frac{at}{\sqrt{(1 + a\xi/c^2)^2 + a^2t^2/c^2}}, \tag{49}$$

where t and x are the Cartesian coordinates used by the inertial observers in I. This velocity clearly depends on the position of the Rindler observers, that is, on ξ. Hence, if we choose to write the metric in terms of proper distances and a proper time for a set of accelerated observers, in general, we will have to choose the proper coordinate system of one of them. For a coordinate system that does not privilege any of the Rindler observers, see Ref. [16]

All the results presented here are valid under the hypothesis of locality. For example, the spectral shift (39) is the same as the one in Ref. [6], Eq. (31), when one neglects the nonlocal effects. For a discussion on this hypothesis and its limitations, see Ref. [7]

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